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AUTHOR(S):

Ikeda, Masahiro; Inui, Takahisa

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# SOME NON-EXISTENCE RESULTS FOR THE SEMILINEAR SCHRÖDINGER EQUATION WITHOUT GAUGE INVARIANCE

MASAHIRO IKEDA AND TAKAHISA INUI

ABSTRACT. We consider the Cauchy problem for the semilinear Schrödinger equation

$$(NLS) \quad \begin{cases} (i\partial_t + \Delta)u = \mu|u|^p, & (t, x) \in [0, T_\lambda] \times \mathbb{R}^d, \\ u(0, x) = \lambda f(x), & x \in \mathbb{R}^d, \end{cases}$$

where  $u = u(t, x)$  is a  $\mathbb{C}$ -valued unknown function,  $\mu \in \mathbb{C} \setminus \{0\}$ ,  $p > 1$ ,  $\lambda \geq 0$ ,  $f = f(x)$  is a  $\mathbb{C}$ -valued given function and  $T_\lambda$  is a maximal existence time of the solution.

Our first aim in the present paper is to prove a large data blow-up result for (NLS) in  $H^s$ -critical or  $H^s$ -subcritical case  $p \leq p_s := 1 + 4/(d - 2s)$ , for some  $s \geq 0$ . More precisely, we show that in the case  $1 < p \leq p_s$ , for a suitable  $H^s$ -function  $f$ , there exists  $\lambda_0 > 0$  such that for any  $\lambda > \lambda_0$ , the following estimates

$$T_\lambda \leq C\lambda^{-\kappa} \quad \text{and} \quad \begin{cases} \lim_{t \rightarrow T_\lambda - 0} \|u(t)\|_{H^s} = \infty, & (\text{if } 1 < p < p_s), \\ \|u\|_{L_t^r(0, T_\lambda; B_{\rho, 2}^s)} = \infty, & (\text{if } p = p_s), \end{cases}$$

hold, where  $\kappa, C > 0$  are constants independent of  $\lambda$  and  $(r, \rho)$  is an admissible pair (see Theorem 2.3).

Our second aim is to prove non-existence local weak-solution for (NLS) in the  $H^s$ -supercritical case  $p > p_s$ , which means that if  $p > p_s$ , then there exists a  $H^s$ -function  $f$  such that if there exist  $T > 0$  and a weak-solution  $u$  to (NLS) on  $[0, T)$ , then  $\lambda = 0$  (see Theorem 2.5).

## 1. INTRODUCTION

We study the Cauchy problem for the semilinear Schrödinger equation

$$(NLS) \quad \begin{cases} (i\partial_t + \Delta)u = F(u), & (t, x) \in [0, T] \times \mathbb{R}^d, \\ u(0, x) = \lambda f(x), & x \in \mathbb{R}^d, \end{cases}$$

where  $T > 0, d \in \mathbb{N}$  is the space dimension,  $u = u(t, x)$  is a  $\mathbb{C}$ -valued unknown function of  $(t, x)$ ,  $F = F(z) : \mathbb{C} \rightarrow \mathbb{C}$  denotes the nonlinearity,  $f = f(x)$  is a  $\mathbb{C}$ -valued prescribed function of  $x$ , and  $\lambda \geq 0$  is a non-negative parameter. Throughout this paper, we assume that the nonlinearity  $F : \mathbb{C} \rightarrow \mathbb{C}$  is continuously differentiable in the real sense and satisfies the following estimates

$$(A) \quad \begin{cases} F(z) = O(|z|^p), & F_z(z), F_{\bar{z}}(z) = O(|z|^{p-1}), \\ F_z(z) - F_z(w), F_{\bar{z}}(z) - F_{\bar{z}}(w) = O(|z - w|^{\min\{p-1, 1\}}(|z| + |w|)^{\max\{0, p-2\}}), \end{cases}$$

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for some  $p \geq 1$ , where  $F_z$  and  $F_{\bar{z}}$  are the complex derivatives

$$F_z := \frac{1}{2} \left( \frac{\partial F}{\partial x} - i \frac{\partial F}{\partial y} \right), \quad F_{\bar{z}} := \frac{1}{2} \left( \frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} \right).$$

The typical examples satisfying the assumptions (A) are the following power type nonlinearities:

$$(1.1) \quad \pm |z|^{p-1} z, \quad \text{and} \quad \pm |z|^p.$$

For these  $p$ -th order nonlinearities, the equation (NLS) is invariant under the scale transformation

$$u(t, x) \mapsto u_\gamma(t, x) := \gamma^{\frac{2}{p-1}} u(\gamma^2 t, \gamma x), \quad \text{for } \gamma > 0.$$

Moreover, the direct computation gives

$$\|u_\gamma(0, \cdot)\|_{\dot{H}^s} = \gamma^{\frac{2}{p-1} - \frac{d-2s}{2}} \|u(0, \cdot)\|_{\dot{H}^s},$$

where  $\dot{H}^s$  denotes the homogeneous Sobolev space. Thus if the order  $p$  satisfies

$$\frac{2}{p-1} - \frac{d-2s}{2} = 0 \quad \Leftrightarrow \quad p = p_s := 1 + \frac{4}{d-2s},$$

then the  $\dot{H}^s$ -norm of the initial data is also invariant. Therefore, the case  $p = p_s$  is called  $H^s$ -critical case. And the case of  $p < p_s$  (resp.  $p > p_s$ ) is called  $H^s$ -subcritical case (resp.  $H^s$ -supercritical case). It is expected that the regularity  $s = s_c := d/2 - 2/(p-1)$  is the threshold between well-posedness and ill-posedness for nonlinear partial differential equations. These phenomena have been verified for several equations, though there are many fundamental PDEs in which it is not known whether these phenomena is valid or not.

We note that though both  $\pm |z|^{p-1} z$  and  $\pm |z|^p$  are the  $p$ -th order, there is one major difference. That is,  $\pm |z|^{p-1} z$  satisfy the gauge invariant property:

$$F(e^{i\theta} z) = e^{i\theta} F(z), \quad \text{for } \theta \in \mathbb{R},$$

though  $\pm |z|^p$  do not. It is quite interesting to investigate the differences of the nonlinear effects on solutions between these nonlinearities. The equation (NLS) with  $F(u) = |u|^p$  is related to the Gross-Pitaevskii equation, which describes the Bose-Einstein condensate in physics. The solution  $\psi$  of the Gross-Pitaevskii equation tends to a non-zero constant as time  $t$  tends to infinity. The term  $|u|^p$  appears if we use the changing variable  $\psi = u + \text{constant}$ . Thus, we expect that the analysis of (NLS) with  $F(u) = |u|^p$  may be helpful for the study of the Gross-Pitaevskii equation (See [22, 46]). There are large amount of literatures about (NLS) with the gauge invariant nonlinearity  $\pm |z|^{p-1} z$ . However, (NLS) with the non-gauge invariant nonlinearity  $\pm |z|^p$  has been less understood. In this paper, we mainly consider (NLS) with the non-gauge invariant nonlinearity  $\pm |z|^p$  and give comparison of the properties of the solution to (NLS) with  $\pm |z|^p$  to those with  $\pm |z|^{p-1} z$ .

Next we recall some previous results about (NLS) with the general nonlinearity  $F$ .

**Case 1. General nonlinearity  $F(z)$  satisfying (A).** We mainly remember the results in the cases  $s = 0$  ( $L^2$ -solution) and  $s = 1$  ( $H^1$ -solution) for simplicity, which are especially important from the physical viewpoints. It is well known that the equation (NLS) is locally well-posed in  $H^s(\mathbb{R}^d)$  in  $H^s$ -subcritical ( $1 \leq p < p_s$ ) or  $H^s$ -critical ( $p = p_s$ ) for arbitrary  $\lambda \geq 0$  and  $f \in H^s(\mathbb{R}^d)$  (see [15, 52, 7, 8, 6] etc.). We note that in the  $H^s$ -subcritical

case, the existence time of local solutions depends only on the  $H^s$ -norm of the initial data, though in the  $H^s$ -critical case, it may depend not only on the norm but on the shape of the data. In the critical case ( $p = p_s$ ), the equation (NLS) is globally well-posed in  $H^s(\mathbb{R}^d)$  for small  $\lambda \geq 0$  and arbitrary  $f \in H^s(\mathbb{R}^d)$  (see [7]). And the global solution tends to a free solution in  $H^s(\mathbb{R}^d)$  as  $t \rightarrow \pm\infty$ , which is called “scattering to a free solution in  $H^s(\mathbb{R}^d)$  as  $t \rightarrow \pm\infty$ ”. The proof of the above results is based on the Strichartz estimates for the linear Schrödinger equation obtained in [48, 16, 34, 55]. For more general  $s \geq 0$ , the similar results were obtained in [38, 7, 8, 6], via the Strichartz estimates and the fractional Leibniz rule in the Sobolev space  $W^{s,p}(\mathbb{R}^d)$  or the Besov space  $B_{p,q}^s(\mathbb{R}^d)$ . In order to prove well-posedness result for higher order regularity  $s \geq 0$ , we need some restrictions on the order  $p$ . We mainly consider the case  $s = 0, 1$  or  $s \in (0, \min(1, d/2))$  for simplicity.

We state the definition of  $H^s$ -solution and local well-posedness results in  $H^s$ -sense precisely.

**Definition 1.1** (admissible pair). *We say that  $(\theta, \varrho)$  is an admissible pair if it satisfies*

$$\frac{2}{\theta} = d \left( \frac{1}{2} - \frac{1}{\varrho} \right)$$

and

$$2 \leq \varrho \leq \frac{2d}{d-2} \quad (2 \leq \varrho \leq \infty \text{ if } d = 1, \quad 2 \leq \varrho < \infty \text{ if } d = 2).$$

We only consider the case  $s = 0, 1$  or  $s \in (0, \min(0, d/2))$  for simplicity and introduce the auxiliary space  $Y_{\theta,\varrho}^s(T)$  as

$$Y_{\theta,\varrho}^s(T) := \begin{cases} L_t^\theta(0, T; W_x^{s,\varrho}(\mathbb{R}^d)), & \text{if } s = 0, 1, \\ L_t^\theta(0, T; B_{\varrho,2}^s(\mathbb{R}^d)). & \text{if } s \in (0, \min(1, 2/d)). \end{cases}$$

**Definition 1.2** ( $H^s$ -solution, its lifespan). *We say that a function  $u : [0, T) \times \mathbb{R}^d \rightarrow \mathbb{C}$  is  $H^s$ -solution to (NLS), if  $u$  lies in the class*

$$X_T^s := C([0, T); H^s(\mathbb{R}^d)) \cap Y_{\gamma,\rho}^s(T),$$

where  $\gamma$  and  $\rho$  are defined by (1.2) below, and obeys the Duhamel formula

$$u(0, x) = \lambda f(x) \text{ and } u(t, x) = e^{it\Delta} u(0, x) - i \int_0^t e^{i(t-t')\Delta} F(u(t', x)) dt',$$

where  $e^{it\Delta}$  is the free Schrödinger evolution group. We also define the lifespan of the solution as

$$T(\lambda) = T(\lambda, f, s) := \sup\{T \in (0, \infty]; \text{ there exists a unique } H^s\text{-solution } u \text{ on } [0, T)\}.$$

We denote the closed ball centered at the origin with the radius  $r$  in  $H^s(\mathbb{R}^d)$  by

$$B_r(H^s) := \{f \in H^s(\mathbb{R}^d); \|f\|_{H^s} \leq r\}.$$

We state a large data local well-posedness result in  $H^s(\mathbb{R}^d)$  in  $H^s$ -subcritical case.

**Theorem 1.3** (L.W.P. in  $H^s$ -subcritical, see [6, 7, 8, 52]). *Let  $d \in \mathbb{N}$ ,  $s = 0, 1$ , or  $s \in (0, \min(1, d/2))$ ,  $1 \leq p < p_s$ . Then the Cauchy problem (NLS) is locally well-posed in  $H^s(\mathbb{R}^d)$  for arbitrary  $u(0, \cdot) \in H^s(\mathbb{R}^d)$ . More precisely, the following statements hold:*

- (Existence) For any  $r > 0$  and for any initial data  $u(0, \cdot) \in B_r(H^s)$ , there exists  $T = T(r) > 0$  such that there exists a  $H^s$ -solution  $u \in X_T^s$  to (NLS) on  $[0, T]$ . Here  $(\gamma, \rho)$  is defined by

$$(1.2) \quad \gamma := \frac{4(p+1)}{(p-1)(d-2s)}, \quad \rho := \frac{d(p+1)}{d+s(p-1)}.$$

Moreover for any admissible pair  $(\theta, \varrho)$ , the solution  $u$  is in  $Y_{\theta, \varrho}^s(T)$ .

- (Uniqueness) Let  $u \in X_T^s$  be the above solution,  $0 < T_1 \leq T$  and  $v \in X_{T_1}^s$  be another  $H^s$ -solution of (NLS) on  $[0, T_1]$ . If  $v(0, \cdot) = u(0, \cdot)$ , then  $v = u|_{[0, T_1]}$ .
- (Continuity of the flow map) The flow map  $B_r(H^s) \mapsto X_T^s$ ,  $u(0, \cdot) \mapsto u$  is Lipschitz continuous.
- (Blow-up criterion) Either

$$(i) \ T(\lambda) = \infty \quad \text{or} \quad (ii) \ T(\lambda) < \infty \quad \text{and} \quad \liminf_{t \rightarrow T(\lambda)-0} \|u(t)\|_{H^s} = \infty$$

is valid.

- (Lower estimate of the lifespan) There exists a positive constant  $c = c(d, s, p, \|f\|_{H^s}) > 0$  such that

$$T(\lambda) \geq c \begin{cases} \lambda^{-1/\omega_0}, & \text{if } \lambda \in (0, 1], \\ \lambda^{-1/\omega_s}, & \text{if } \lambda \in (1, \infty), \end{cases}$$

where

$$\omega_s := \frac{1}{p-1} - \frac{d-2s}{4}.$$

The similar statements also hold in the negative time direction.

**Remark 1.1.** In the  $H^s$ -subcritical case, the maximal existence time  $T$  of the solution to (NLS) depends only on the  $H^s$ -norm of the initial value.

We also recall a large data local well-posedness and small data global well-posedness result in  $H^s$ -sense in the  $H^s$ -critical case.

**Corollary 1.4** (L.W.P. for arbitrary data and G.W.P. for small data in  $H^s$ -sense in the  $H^s$ -critical case, see [6, 7, 8]). Let  $d \in \mathbb{N}$ ,  $s = 0, 1$ , or  $s \in (0, \min(1, d/2))$  and  $p = p_s$ . Then the Cauchy problem (NLS) is locally well-posed in  $H^s(\mathbb{R}^d)$  for arbitrary  $\lambda \geq 0$  and  $f \in H^s(\mathbb{R}^d)$ . More precisely, the following statements hold:

- (Existence) For any  $r > 0$  and  $u(0, \cdot) \in B_r(H^s)$ , there exists  $T = T(r, u(0, \cdot)) > 0$  such that there exists a  $H^s$ -solution  $u \in X_T^s$  to (NLS) on  $[0, T]$ .
- (Uniqueness) Let  $u \in X_T^s$  be the above solution of (NLS),  $0 < T_1 \leq T$  and  $v \in X_{T_1}^s$  be another solution of (NLS) on  $[0, T_1]$ . If  $v(0, \cdot) = u(0, \cdot)$ , then  $v = u|_{[0, T_1]}$ .
- (Continuity of the flow map) The flow map  $B_r(H^s) \mapsto X_T^s$ ,  $u(0, \cdot) \mapsto u$  is Lipschitz continuous.
- (Blow-up criterion) Either

$$(i) \ T(\lambda) = \infty \quad \text{or} \quad (ii) \ T(\lambda) < \infty \quad \text{and} \quad \|u\|_{Y_{\gamma, \rho}^s(T(\lambda))} = \infty$$

is valid.

- (G.W.P. for small data) There exists a constant  $\lambda_1 = \lambda_1(d, s) > 0$  such that for any  $\lambda \in [0, \lambda_1]$  and  $f \in H^s(\mathbb{R}^d)$ , the above statements are valid with  $T = \infty$ . Moreover, the global solution  $u : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{C}$  scatters to a free solution in  $H^s(\mathbb{R}^d)$  as  $t \rightarrow +\infty$ .

The similar statements are valid in the negative time direction.

**Remark 1.2.** In the critical case  $p = p_s$ , the similar result as in Corollary 1.4 holds, even if the inhomogeneous spaces  $H^s(\mathbb{R}^d)$ ,  $W^{s,p}(\mathbb{R}^d)$  and  $B_{p,2}^s(\mathbb{R}^d)$  are replaced by the homogeneous spaces  $\dot{H}^s(\mathbb{R}^d)$ ,  $\dot{W}^{s,p}(\mathbb{R}^d)$  and  $\dot{B}_{p,2}^s(\mathbb{R}^d)$  respectively.

**Case 2. Gauge invariant type nonlinearity**  $F(z) = \pm|z|^{p-1}z$ . In this case, the equation (NLS) has several conserved quantities. For example, for a solution  $u$ , the mass, the energy and the momentum conservation laws hold:

$$(\text{mass}) \quad M[u](t) := \|u(t)\|_{L^2} = M[u](0),$$

$$(\text{energy}) \quad E[u](t) := \|\nabla u(t)\|_{L^2}^2 \mp \frac{2}{p+1} \|u(t)\|_{L^{p+1}}^{p+1} = E[u](0),$$

$$(\text{momentum}) \quad P[u](t) := \Im \int_{\mathbb{R}^d} \bar{u} \nabla u dx = P[u](0),$$

for any  $t \in [0, T)$ . Moreover, the pseudoconformal conservation law is also valid:

$$(1.3) \quad \frac{d^2}{dt^2} \| |\cdot| u(t) \|_{L^2}^2 = 16E[u](0) \pm \frac{4d(p-1-4/d)}{p+1} \|u(t)\|_{L^{p+1}}^{p+1},$$

for  $H^1$ -solution  $u$  with  $|\cdot|u(0, \cdot) \in L^2(\mathbb{R}^d)$ .

**Subcase 2-1.  $L^2$ -solution.** The equation (NLS) is globally well-posed in  $L^2(\mathbb{R}^d)$  for arbitrary  $L^2(\mathbb{R}^d)$  initial data in  $L^2$ -subcritical case ( $1 \leq p < p_0$ ), which follows from the fact that the existence time of local solutions depends only on the size of the initial data, and the a-priori  $L^2$ -bound of solutions via the above mass conservation law. In the  $L^2$ -critical case ( $p = p_0$ ), several types of solutions appear, corresponding to the sign of the coefficient of the nonlinearity and the size of the initial data. In the defocusing case ( $F(z) = |z|^{4/d}z$ ), it was proved in [11, 12, 13] that the equation (NLS) is globally well-posed in  $L^2(\mathbb{R}^d)$  for arbitrary  $L^2(\mathbb{R}^d)$  initial data, and the solution scatters to a free solution in  $L^2(\mathbb{R}^d)$  as  $t \rightarrow \pm\infty$  (see also [36] for the radially symmetric case). In the focusing case ( $F(z) = -|z|^{4/d}z$ ), the function  $e^{it}Q(x)$  satisfies (NLS) for any  $t \in \mathbb{R}$ , if  $\lambda f = Q$ , where  $Q = Q(x)$  is called the grand state and is the non-negative solution of the elliptic equation

$$(1.4) \quad \Delta Q - Q + Q^p = 0, \quad x \in \mathbb{R}^d.$$

The existence and uniqueness of  $Q$  were established in [3] and [39] respectively. If  $u(t, x) = e^{it}Q(x)$ , then  $\|u(t)\|_{L^2} = \|Q\|_{L^2} < \infty$  for any  $t \in \mathbb{R}$ , but the space-time norm is not bounded:  $\|u\|_{Y_{s,\rho}^s(\pm\infty)} = \infty$ . Moreover, by transforming this to

$$u(t, x) = (T - t)^{-d/2} \exp \left( i \frac{t}{T - t} - i \frac{|x|^2}{4(T - t)} \right) Q \left( \frac{x}{T - t} \right),$$

we can find a blow-up solution in a finite time. On the other hand, it was proved in [14] that if  $\|u(0, \cdot)\|_{L^2} < \|Q\|_{L^2}$ , then the equation (NLS) with  $F(z) = -|z|^{4/d}z$  is globally well-posed

in  $L^2(\mathbb{R}^d)$  and the solution scatters to a free solution in  $L^2(\mathbb{R}^d)$ . For more information about  $L^2$ -solution, see [5, 35].

**Subcase 2-2.  $H^1$ -solution.** For  $H^1$ -solution, the situations are more complicated than  $L^2$ -solution. In the defocusing and  $H^1$ -subcritical ( $1 \leq p < p_1$ ) case, the equation (NLS) is globally well-posed in  $H^1(\mathbb{R}^d)$  for arbitrary  $H^1(\mathbb{R}^d)$ -initial data, which follows from an a-priori bound of  $H^1$ -norm via the mass and energy conservation laws. The asymptotic behaviors of the global  $H^1$ -solutions were also studied in the papers [2, 17, 40, 54]. In the defocusing and  $H^1$ -critical case ( $F(z) = |z|^{\frac{4}{d-2}}z$ ), it was proved in [10, 45, 53] that the equation (NLS) is globally well-posed in  $\dot{H}^1(\mathbb{R}^d)$  for arbitrary data with finite energy  $E[u](0) < \infty$ , and the  $\dot{H}^1$ -solution scatters in the energy space  $\dot{H}^1(\mathbb{R}^d)$  (see also [4, 20, 49] for radially symmetric case). On the other hand, in the focusing case ( $F(z) = -|z|^{\frac{4}{d-2}}z$ ), the Talenti function

$$W(x) := \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}} \in \dot{H}^1(\mathbb{R}^d)$$

satisfies the stationary equation

$$\Delta u = -|u|^{\frac{4}{d-2}}u = -|u|^{p_1}, \quad x \in \mathbb{R}^d$$

and we can see that  $W$  is a global  $H^1$ -solution to (NLS) with  $\lambda f = W$ , but the time-space norm  $\|\cdot\|_{L_t^\gamma(\mathbb{R}; \dot{W}^{1,p}(\mathbb{R}^d))}$  is not finite. The scattering results also were obtained for  $H^1$ -solution, (see [32, 37]).

**Case 3. Non-gauge invariant type nonlinearity**  $F(z) = |z|^p$ . In this case, it is not trivial whether the conservation laws hold or not, different from the case of the gauge invariant nonlinearity  $\pm|z|^{p-1}z$ . In fact, it was proved in [29] that the equation (NLS) does not have the mass conservation law. Moreover, in our previous paper [26], it was shown that there exists a blow-up  $H^s$ -solution (some  $s \geq 0$ ) for the equation (NLS) in the  $L^2$ -subcritical case ( $1 < p < p_0$ ), even if  $\lambda > 0$  is small (see Theorem 1.5 below). In the (NLS) with  $F(z) = |z|^p$ , the energy conservation law does not seem to hold, though the momentum is conserved.

Next we recall our previous result obtained in [26], which gives existence of a blow-up  $H^s$ -solution to (NLS) for a suitable  $f \in H^s(\mathbb{R}^d)$  and arbitrary  $\lambda > 0$  and an upper estimate of the lifespan for small  $\lambda$ :

**Theorem 1.5** (Blow-up  $H^s$ -solution for arbitrary  $\lambda > 0$  and suitable  $f \in H^s(\mathbb{R}^d)$  in the  $L^2$ -subcritical case, see [26]). *Let  $d \in \mathbb{N}$ ,  $s = 0, 1$  or  $s \in (0, \min(1, d/2))$ ,  $1 < p < p_0$ ,  $\mu \in \mathbb{C} \setminus \{0\}$ ,  $F(z) = \mu|z|^p$  and  $f \in H^s(\mathbb{R}^d)$ . We assume that the function  $f$  satisfies*

$$(1.5) \quad (\Im \mu)(\Re f)(x) \quad \text{or} \quad -(\Re \mu)(\Im f)(x) \geq \begin{cases} |x|^{-k}, & (|x| > 1), \\ 0, & (|x| \leq 1), \end{cases}$$

where  $d/2 < k < 2/(p-1)$ . Then there exists a constant  $C > 0$  depending only on  $d, p, k, |\mu|$  such that for any  $\lambda \in (0, 1]$ , the following estimates hold:

$$(1.6) \quad T(\lambda) \leq C\lambda^{-1/\kappa}, \quad \liminf_{t \rightarrow T(\lambda)-0} \|u(t)\|_{H^s} = \infty,$$



where  $\kappa := 1/(p-1) - k/2$ . The similar statement also holds for the negative time direction if the right hand side of (1.5) is replaced by

$$-(\Im \mu)(\Re f)(x) \quad \text{or} \quad (\Re \mu)(\Im f)(x).$$

However, the following three problems have been open as far as the authors know:

- Problem 1.6.** (1) Can we obtain any upper estimates of the lifespan such as (1.6) when  $\lambda(\geq 1)$  is large? In Theorem 1.5, the upper estimate (1.6) is valid only in the small  $\lambda \in (0, 1]$  case. We give an affirmative answer to this question in Theorem 2.3.
- (2) In the  $H^s$ -critical ( $p = p_s$ ) and large data ( $\lambda \gg 1$ ) case, does local  $H^s$  (or  $\dot{H}^s$ )-solution (for some  $s \geq 0$ ) obtained in [7] exist globally or not? We prove existence of a blow-up  $H^s$ -solution in this case in Corollary 2.3.
- (3) In the  $H^s$ -supercritical case ( $p > p_s$ ), if the initial data belongs to  $H^s(\mathbb{R}^d)$ , then does there exist any local solutions to (NLS)? There have been no results about this issue except for  $p = 2$  or 4. We give a non-existence result in Theorem 2.5.

Next we recall some other related results to the above questions. In the case  $p = 2$ , since the nonlinearity  $F(z)$  can be written as  $F(z) = |z|^2 = z\bar{z}$ , we can see that the Fourier analysis is effective. In fact, by using the Fourier restriction norm method, low-regularity L.W.P. in  $H^s(\mathbb{R}^d)$  with  $s > -\frac{1}{4}$  was obtained in [33] ( $d = 1$ ), [9] ( $d = 2$ ), and [50] ( $d = 3$ ). In the case  $d = 1, p = 4$ , L.W.P. in  $H^s(\mathbb{R}^d)$  with  $s > -\frac{1}{8}$  was obtained in [21]. On the other hand, ill-posedness to (NLS) has not been less understood, though in the case  $F(z) = c_1 z^2 + c_2 \bar{z}^2$  for some  $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ , ill-posedness results were studied in [1, 9, 38]. For more information about (NLS) with  $F(z) = |z|^p$ , see [18, 23, 24, 25, 43, 44].

At the end of this section, we introduce some function spaces and notations used throughout this paper. For  $1 \leq p \leq \infty$ , we denote the Lebesgue space by  $L^p = L^p(\mathbb{R}^l)$ , where  $l = 1, d$ , or  $1 + d$  depend on the context, with the norm  $\|f\|_{L^p} := (\int_{\mathbb{R}^l} |f(x)|^p dx)^{1/p}$  if  $1 \leq p < \infty$  and  $\|f\|_{L^\infty} := \text{ess.sup}_{x \in \mathbb{R}^l} |f(x)|$ . For a time interval  $I$  and a Banach space  $X$ , we use the time-space Lebesgue space  $L_t^p(I; X)$ , with the norm  $\|u\|_{L_t^p(I; X)} := \| \|u(t)\|_X \|_{L_t^p(I)}$ . We often omit the time interval  $I$  and denote  $L_t^p X = L_t^p(I; X)$ , if it does not cause a confusion. We denote by  $L_{loc}^p(I \times \mathbb{R}^d)$  the set of measurable functions  $f : I \times \mathbb{R}^d \mapsto \mathbb{C}$  such that for every compact interval  $J \subset I \times \mathbb{R}^d$ ,  $f|_J \in L^p(J)$ . Let  $\mathcal{S}(\mathbb{R}^l)$  be the rapidly decaying function space. For  $f \in \mathcal{S}(\mathbb{R}^d)$ , we define the Fourier transform of  $f$  by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx,$$

and the inverse Fourier transform of  $f$  by

$$\mathcal{F}^{-1}[f](x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(\xi) d\xi,$$

and extend them to  $\mathcal{S}'(\mathbb{R}^d)$  by duality. We define the inhomogeneous Sobolev spaces by  $W^{s,p}(\mathbb{R}^d)$  with the norm  $\|f\|_{W^{s,p}} := \|\mathcal{F}^{-1}[\langle \xi \rangle^s \hat{f}]\|_{L_x^p}$ , where  $\langle \cdot \rangle := 1 + |\cdot|$ . We also use  $H^s(\mathbb{R}^d) := W^{s,2}(\mathbb{R}^d)$ . We denote the homogeneous Sobolev space by  $\dot{H}^s(\mathbb{R}^d)$  with the norm  $\|f\|_{\dot{H}^s} := \| |\xi|^s \hat{f} \|_{L_\xi^2}$ . We introduce a test-function  $\varphi \in C_0^\infty(\mathbb{R}^d)$  satisfying  $\varphi(\xi) = 1$ , if  $|\xi| \leq 1$  and  $\varphi(\xi) = 0$ , if  $|\xi| \geq 2$ . We define the Littlewood-Paley decomposition  $\{\varphi_j\}_{j \in \mathbb{Z}} \subset C_0^\infty(\mathbb{R}^d)$



by  $\varphi_j(\xi) := \varphi\left(\frac{\xi}{2^j}\right) - \varphi\left(\frac{\xi}{2^{j-1}}\right)$ . For  $s \in \mathbb{R}$  and  $1 \leq p, q \leq \infty$ , we define the inhomogeneous Besov space  $B_{p,q}^s(\mathbb{R}^d)$  as  $B_{p,q}^s(\mathbb{R}^d) := \{f \in \mathcal{S}' ; \|f\|_{B_{p,q}^s} < \infty\}$ , with the norm

$$\|f\|_{B_{p,q}^s} := \|\mathcal{F}^{-1}(\varphi \hat{f})\|_{L^p} + \begin{cases} \left( \sum_{j=1}^{\infty} (2^{sj} \|\mathcal{F}^{-1}(\varphi_j \hat{f})\|_{L^p})^q \right)^{1/q}, & \text{if } q < \infty, \\ \sup_{j \geq 1} 2^{sj} \|\mathcal{F}^{-1}(\varphi_j \hat{f})\|_{L^p}, & \text{if } q = \infty. \end{cases}$$

## 2. MAIN RESULT

In this section, we state our main results in this paper. We reduce our problems with respect to  $H^s$ -solution into existence of the weak-solution defined as follows.

**Definition 2.1** (Weak-solution, its lifespan). *We say that  $u$  is a weak-solution to (NLS) on  $[0, T)$ , if  $u$  belongs to  $L_{loc}^p([0, T) \times \mathbb{R}^d)$  and the following identity*

$$(2.1) \quad \begin{aligned} & \int_{[0,T) \times \mathbb{R}^d} u(t, x) \{ -i(\partial_t \psi)(t, x) + (\Delta \psi)(t, x) \} dx dt \\ & = i\lambda \int_{\mathbb{R}^d} f(x) \psi(0, x) dx + \mu \int_{[0,T) \times \mathbb{R}^d} F(u(t, x)) \psi(t, x) dx dt \end{aligned}$$

holds for any  $\psi \in C_0^\infty([0, T) \times \mathbb{R}^d)$ . We denote the lifespan for the weak solution by

$$T_w(\lambda) := \sup\{T \in (0, \infty]; \text{ there exists a unique weak-solution } u \text{ to (NLS)}\}.$$

Next we state a non-existence result for the global weak-solution for  $p > 1$  for a suitable  $f$  and large  $\lambda$ .

**Proposition 2.2** (Non-existence of the global weak-solution for  $p > 1$  and large data). *Let  $d \in \mathbb{N}$ ,  $p > 1$ ,  $\mu \in \mathbb{C} \setminus \{0\}$ ,  $F(z) = \mu|z|^p$ ,  $f \in L_{loc}^1(\mathbb{R}^d)$  and  $u$  be a weak solution on  $[0, T_w(\lambda))$ . We assume that the function  $f$  satisfies*

$$(2.2) \quad (\Im \mu)(\Re f)(x) \quad \text{or} \quad -(\Re \mu)(\Im f)(x) \geq \begin{cases} |x|^{-k}, & (|x| \leq 1), \\ 0, & (|x| > 1), \end{cases}$$

where  $k < \min\left(d, \frac{2}{p-1}\right)$ . Then there exist constants  $\lambda_0 > 0$  and  $C > 0$  depending only on  $d, p, k, |\mu|$  such that for any  $\lambda > \lambda_0$ ,

$$(2.3) \quad T_w(\lambda) \leq C\lambda^{-1/\kappa},$$

where  $\kappa := \frac{1}{p-1} - \frac{k}{2}$ . The similar statement also holds for the negative time direction if the right hand side of (2.2) is replaced by

$$(2.4) \quad -(\Im \mu)(\Re f)(x) \quad \text{or} \quad (\Re \mu)(\Im f)(x).$$

Next we give a large data blow-up result for  $H^s$ -solution in the  $H^s$ -subcritical or critical case  $1 < p \leq p_s$ .

**Theorem 2.3** (Large data blow-up result for  $H^s$ -solution in the  $H^s$ -critical or subcritical case). *Let  $d \in \mathbb{N}$ ,  $s = 0, 1$ , or  $s \in (0, \min(1, d/2))$ ,  $1 < p \leq p_s$ ,  $\mu \in \mathbb{C} \setminus \{0\}$ ,  $F(z) = \mu|z|^p$  and  $f \in H^s(\mathbb{R}^d)$ . We assume that the function  $f$  satisfies (2.2) with  $k < d/2 - s$ . Then there exist constants  $\lambda_0 > 0$  and  $C > 0$  depending only on  $d, p, k, |\mu|$  such that for any  $\lambda > \lambda_0$ ,*

$$(2.5) \quad T(\lambda) \leq C\lambda^{-1/\kappa}$$

where  $\kappa := 1/(p-1) - k/2$ . Moreover, the norm of the solution blows up at  $t = T(\lambda)$ :

$$(2.6) \quad \liminf_{t \rightarrow T(\lambda)-0} \|u(t)\|_{H^s} = \infty, \quad \text{if } p < p_s, \quad \|u\|_{Y_{\gamma,\rho}^s(T(\lambda))} = \infty, \quad \text{if } p = p_s,$$

where  $(\gamma, \rho)$  is defined by (1.2). The similar statement also holds for the negative time direction if the right hand side of (2.2) is replaced by (2.4).

- Remark 2.1.** (1) In the critical case  $p = p_s$ , the similar statement also holds, even if the inhomogeneous spaces  $H^s(\mathbb{R}^d)$ ,  $W^{s,p}(\mathbb{R}^d)$  and  $B_{\rho,2}^s(\mathbb{R}^d)$  are replaced by the homogeneous spaces  $\dot{H}^s(\mathbb{R}^d)$ ,  $\dot{W}^{s,p}(\mathbb{R}^d)$  and  $\dot{B}_{\rho,2}^s(\mathbb{R}^d)$  respectively.
- (2) We note that in Theorem 1.5, in order to prove a small data blow-up result, we choose the function  $f$  whose imaginary part has a positive value larger than  $|x|^{-k}$  for large  $|x|$ , though in Theorem 2.3, in order to show the large data blow-up result, we take the function  $f$  whose imaginary part has a singularity at the origin  $x = 0$ .

Next we state a non-existence result for the local weak-solution for  $p > p_F$  for suitable  $f$  and arbitrary  $\lambda > 0$ , where  $p_F := 1 + 2/d$  is called the Fujita exponent.

**Proposition 2.4** (Non-existence of the local weak-solution in the case  $p > p_F$  and  $L_{loc}^1$ -data). *Let  $d \in \mathbb{N}$ ,  $p > p_F$ ,  $\mu \in \mathbb{C} \setminus \{0\}$ ,  $F(z) = \mu|z|^p$ ,  $\lambda \geq 0$  and  $f \in L_{loc}^1(\mathbb{R}^d)$ . We assume that the function  $f$  satisfies the estimate (2.2) with  $\frac{2}{p-1} < k < d$ . Then if there exist  $T > 0$  and a weak-solution  $u$  to (NLS) on  $[0, T)$ , then  $\lambda = 0$ . The similar statement also holds for the negative time direction if the right hand side of (2.2) is replaced by (2.4).*

Next we give a non-existence result for the local weak-solution in the  $H^s$ -supercritical case for suitable  $H^s$ -function  $f$  for arbitrary  $\lambda > 0$ .

**Theorem 2.5** (Non-existence of the local weak-solution in  $H^s$ -supercritical for suitable  $H^s(\mathbb{R}^d)$ -data). *Let  $d \in \mathbb{N}$ ,  $s < d/2$ ,  $p > p_s$ ,  $\mu \in \mathbb{C} \setminus \{0\}$ ,  $F(z) = \mu|z|^p$ ,  $\lambda \geq 0$  and  $f \in H^s(\mathbb{R}^d)$ . We assume that the function  $f$  satisfies (2.2) with  $k \in (2/(p-1), \min(d/2 - s, d))$ . Then, if there exist  $T > 0$  and a weak-solution  $u$  to (NLS) on  $[0, T)$ , then  $\lambda = 0$ . The similar statement also holds for the negative time direction if the right hand side of (2.2) is replaced by (2.4).*

- Remark 2.2.** (1) It should be checked that there exists a function  $f : \mathbb{R}^d \rightarrow \mathbb{C}$  satisfying the assumptions in Theorem 2.5, i.e.  $f \in H^s(\mathbb{R}^d)$  and the estimate (2.2) with  $k \in (2/(p-1), \min(d/2 - s, d))$ . This is proved in Section 5.
- (2) In Theorem 2.5, when  $s < -\frac{d}{2}$ ,  $\min(\frac{d}{2} - s, d) = d$ . Therefore such  $k$  exists if and only if  $p > p_F(> p_s)$ . In this case, we can take the function  $f$  as the Dirac delta function centered at 0.

At the end of this section, we give a few remarks on our main results.

- (1) We recall the definition of the Strauss exponent  $p_S$ :

$$p_S = p_S(d) := \frac{2 - d + \sqrt{d^2 + 12d + 4}}{2d},$$

which often appears in the study of not only NLS but also nonlinear wave equations. We note the relation:

$$p_F < p_S < p_0 < p_1.$$

It is well known that for the nonlinearity  $F$  satisfying (A) with  $p_S < p \leq p_s$ , the equation (NLS) is G.W.P. in  $H^s(\mathbb{R}^d)$  for small initial data in  $H^s(\mathbb{R}^d) \cap L^{1+1/p}(\mathbb{R}^d)$ , and the global  $H^s$ -solution scatters to a free solution in  $H^s(\mathbb{R}^d)$  as  $t \rightarrow \pm\infty$  (see for example Section 6.3 in [6], [47]). On the other hand, it was proved in [26] that for the nonlinearity  $F(z) = |z|^p$  with the opposite case  $1 < p < p_S$ , there exists a blowing-up  $H^s$ -solution to (NLS) for suitable  $f \in H^s(\mathbb{R}^d) \cap L^{1+1/p}(\mathbb{R}^d)$  and even for small  $\lambda > 0$ . However, in the critical case  $p = p_S$ , it was not known whether the local  $H^s$ -solution exists globally or not, even if the data is sufficiently small and smooth. Especially, in the case  $d = 3, p = p_S = 2$ , almost global solution to (NLS) with  $F(z) = |z|^2$  was proved for small and smooth data in [18]. Our Theorem 2.3 gives that in the critical case  $p = p_S$ , there exists a blowing-up  $H^s$ -solution to (NLS) with  $F(z) = |z|^p$  for a suitable  $f \in H^s(\mathbb{R}^d) \cap L^{1+1/p}(\mathbb{R}^d)$  and large  $\lambda > 0$ .

- (2) The ill-posedness results obtained in [33] ( $F(z) = |z|^2$ ) and in [1] ( $F(z) = z^2$ ) imply the irregularity of the flow map. On the other hand, our ill-posedness result (Theorem 2.5) means non-existence of local weak-solution, it should be distinguished from theirs.

### 3. INTEGRAL INEQUALITIES VIA A TEST-FUNCTION METHOD

In this section, we derive two useful inequalities (Lemmas 3.1, 3.2 below) by using suitable test-functions. Though the similar results as Lemma 3.1 were obtained in [56, 38, 29, 28, 26, 27], Lemma 3.2 does not appear in the papers.

We take the two functions  $\eta = \eta(t) \in C_0^\infty([0, \infty))$ ,  $\phi = \phi(x) \in C_0^\infty(\mathbb{R}^d)$  such as  $0 \leq \eta, \phi \leq 1$  and

$$\eta(t) := \begin{cases} 1 & (0 \leq t < 1/2), \\ \text{smooth} & (1/2 \leq t \leq 1), \\ 0 & (t \geq 1), \end{cases} \quad \phi(x) := \begin{cases} 1 & (0 \leq |x| < 1/2), \\ \text{smooth} & (1/2 \leq |x| \leq 1), \\ 0 & (|x| \geq 1). \end{cases}$$

For a parameter  $\tau > 0$ , we also define the time-space function

$$\psi_\tau = \psi_\tau(t, x) := \eta_\tau(t)\phi_\tau(x) := \eta(t/\tau)\phi(x/\sqrt{\tau}).$$

We denote the open ball of radius  $r > 0$  at the origin in  $\mathbb{R}^d$  by  $B(r) := \{x \in \mathbb{R}^d; |x| < r\}$ .

**Lemma 3.1.** *Let  $d \in \mathbb{N}$ ,  $p > 1$ ,  $l \in \mathbb{N}$  with  $l \geq 2q + 1$ ,  $\lambda \geq 0$ ,  $\mu \in \mathbb{C} \setminus \{0\}$ ,  $F(z) = \mu|z|^p$ ,  $f \in L_{loc}^1(\mathbb{R}^d)$  and  $u$  be a weak-solution of (NLS) on  $[0, T)$ . Then there exists a constant  $C > 0$  depending only on  $d, p$  and  $l$ , such that the estimates*

$$(3.1) \quad \begin{cases} -\lambda \int_{B(\sqrt{\tau})} \Im f(x) \phi_\tau^l(x) dx \leq C |\Re \mu|^{-1/(p-1)} \tau^{(d+2)/2-q}, & \text{if } \Re \mu > 0, \\ \lambda \int_{B(\sqrt{\tau})} \Im f(x) \phi_\tau^l(x) dx \leq C |\Re \mu|^{-1/(p-1)} \tau^{(d+2)/2-q}, & \text{if } \Re \mu < 0, \\ \lambda \int_{B(\sqrt{\tau})} \Re f(x) \phi_\tau^l(x) dx \leq C |\Im \mu|^{-1/(p-1)} \tau^{(d+2)/2-q}, & \text{if } \Im \mu > 0, \\ -\lambda \int_{B(\sqrt{\tau})} \Re f(x) \phi_\tau^l(x) dx \leq C |\Im \mu|^{-1/(p-1)} \tau^{(d+2)/2-q}, & \text{if } \Im \mu < 0, \end{cases}$$

are true, for any  $\tau \in (0, T)$ , where  $q := p/(p-1)$ .

*Proof of Lemma 3.1.* We introduce two positive functions of  $\tau \in (0, T)$

$$(3.2) \quad I(\tau) := \int_{[0, \tau) \times B(\sqrt{\tau})} |u(t, x)|^p \psi_\tau^l(t, x) dx dt, \quad J(\tau) := \int_{B(\sqrt{\tau})} f(x) \phi_\tau^l(x) dx.$$

Since  $u$  is a weak-solution on  $[0, T)$  and  $\psi_\tau^l \in C_0^\infty([0, T) \times \mathbb{R}^d)$ , by substituting the test function in Definition 2.1 into  $\psi_\tau^l$ , we have

$$(3.3) \quad \mu I(\tau) + i\lambda J(\tau) = \int_{[0, \tau) \times B(\sqrt{\tau})} (-iu) \partial_t(\psi_\tau^l) dx dt + \int_{[0, \tau) \times B(\sqrt{\tau})} (-u) \Delta(\psi_\tau^l) dx dt.$$

We only consider the case  $\Re \mu > 0$ . The other cases can be proved in the almost similar manner. Taking the real part of the identity (3.3), we obtain

$$(3.4) \quad \Re \mu I(\tau) - \lambda \Im J(\tau) = \int_{[0, \tau) \times B(\sqrt{\tau})} (\Im u) \partial_t(\psi_\tau^l) dx dt + \int_{[0, \tau) \times B(\sqrt{\tau})} (-\Re u) \Delta(\psi_\tau^l) dx dt := K_1 + K_2.$$

We will estimate  $K_1$ . Due to  $l/q - 1 \geq 0$  and the Hölder inequality, we have

$$(3.5) \quad \begin{aligned} K_1 &\leq l\tau^{-1} \int_{[0, \tau) \times B(\sqrt{\tau})} |u| \eta_\tau^{l-1} \phi_\tau^l |\eta'(t/\tau)| dx dt \leq C\tau^{-1} \int_{[0, \tau) \times B(\sqrt{\tau})} |u| \psi_\tau^{l/p} dx dt \\ &\leq C\tau^{-1} \{I(\tau)\}^{1/p} \left( \int_{[0, \tau) \times B(\sqrt{\tau})} dx dt \right)^{1/q} = C\tau^{(d+2)/2q-1} \{I(\tau)\}^{1/p}. \end{aligned}$$

Next we consider  $K_2$ . By  $l/q - 2 \geq 0$ , the Hölder inequality, we obtain

$$(3.6) \quad \begin{aligned} K_2 &\leq l(l-1)\tau^{-1} \int_{[0, \tau) \times B(\sqrt{\tau})} |u| \eta_\tau^l \phi_\tau^{l-2} |(\nabla \phi)(x/\sqrt{\tau})|^2 dx dt \\ &\quad + l\tau^{-1} \int_{[0, \tau) \times B(\sqrt{\tau})} |u| \eta_\tau^l \phi_\tau^{l-1} |(\Delta \phi)(x/\sqrt{\tau})| dx dt \\ &\leq C\tau^{-1} \int_{[0, \tau) \times B(\sqrt{\tau})} |u| \psi_\tau^{l/p} dx dt \leq C\tau^{(d+2)/2q-1} \{I(\tau)\}^{1/p}. \end{aligned}$$

By combining the estimates (3.3)-(3.6), we obtain

$$(3.7) \quad -\lambda \Im J(\tau) \leq C\tau^{(d+2)/2q-1} \{I(\tau)\}^{1/p} - (\Re \mu) I(\tau).$$

We note that since  $p, q > 1$  and  $1/p + 1/q = 1$ , we have  $ab \leq a^p/p + b^q/q$  for  $a, b > 0$ . Thus we can estimate for the first term of the right hand side as

$$(3.8) \quad C\tau^{(d+2)/2q-1} \{I(\tau)\}^{1/p} \leq C(\Re \mu)^{-1/(p-1)} \tau^{(d+2)/2-q} + (\Re \mu) I(\tau).$$

By the estimates (3.7)-(3.8), we have

$$(3.9) \quad -\lambda \Im J(\tau) \leq C(\Re \mu)^{-1/(p-1)} \tau^{(d+2)/2-q},$$

where  $C$  is a positive constant dependent only on  $d, p$  and  $l$ , which completes the proof of the lemma.  $\square$

**Lemma 3.2.** *We assume the same assumptions as in Lemma 3.1. Furthermore we assume that the function  $f$  satisfies (2.2) with  $k \in \mathbb{R}$ . Then the estimate*

$$(3.10) \quad \lambda \leq C\tau^{(k+2)/2-q} \left( \int_{|x| \leq 1/\sqrt{\tau}} |x|^{-k} \phi^l(x) dx \right)^{-1} \times \begin{cases} |\Re \mu|^{\frac{p-2}{p-1}}, & \text{if } \Re \mu \neq 0, \\ |\Im \mu|^{\frac{p-2}{p-1}}, & \text{if } \Im \mu \neq 0, \end{cases}$$

are true, for any  $\tau \in (0, T)$ , where  $C > 0$  is the same constant as in Lemma 3.1.

*Proof of Lemma 3.2.* We only consider the case  $\Re\mu > 0$ . The other cases can be treated in the almost similar manner. Then by (2.2),  $\Im f$  satisfies

$$(3.11) \quad -\Im f(x) \geq \begin{cases} (\Re\mu)^{-1}|x|^{-k}, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1. \end{cases}$$

We consider the lower bound of  $-\Im J(\tau)$ , where  $J(\tau)$  is defined by (3.2). By changing variables and (3.11), we have

$$(3.12) \quad -\Im J(\tau) = \tau^{d/2} \int_{\mathbb{R}^d} -\Im f(\sqrt{\tau}x) \phi^l(x) dx \geq (\Re\mu)^{-1} \tau^{(d-k)/2} \int_{|x| \leq 1/\sqrt{\tau}} |x|^{-k} \phi^l(x) dx$$

for any  $\tau \in (0, T)$ . By combining Lemma 3.1 and (3.12), we obtain (3.10), which completes the proof of the lemma.  $\square$

#### 4. PROOF OF THE MAIN THEOREMS

First we give a proof of Proposition 2.2.

*Proof of Proposition 2.2.* We only consider the case  $\Re\mu \neq 0$ . The other case can be treated in the similar manner. By Lemma 3.2, we have

$$(4.1) \quad \lambda \leq C_1 |\Re\mu|^{\frac{p-2}{p-1}} \tau^{(k+2)/2-q} \{L(\tau)\}^{-1},$$

for any  $\tau \in (0, T_w(\lambda))$ , where

$$(4.2) \quad L(\tau) := \int_{|x| \leq 1/\sqrt{\tau}} |x|^{-k} \phi^l(x) dx.$$

**Claim.** There exists  $\lambda_0 > 0$  depending only on  $d, k$  and  $|\mu|$  such that if  $\lambda > \lambda_0$ , then

$$(4.3) \quad T_w(\lambda) \leq 4.$$

Indeed, on the contrary, we assume that  $T_w(\lambda) > 4$ . Then by (4.1) with  $\tau = 4$ ,

$$(4.4) \quad \lambda \leq C_1 |\Re\mu|^{\frac{p-2}{p-1}} 4^{(k+2)/2-q} \{L(4)\}^{-1}.$$

By changing variables and  $k < d$ , we have

$$(4.5) \quad L(4) = \int_{|x| \leq 1/2} |x|^{-k} dx = C \int_0^{1/2} r^{-k+d-1} dr =: C_2 (< \infty).$$

By combining (4.4)-(4.5), we obtain

$$\lambda \leq C_1 C_2^{-1} 4^{(k+2)/2-q} |\Re\mu|^{\frac{p-2}{p-1}} =: \lambda_0,$$

which leads to a contradiction to  $\lambda > \lambda_0$ . Thus the claim is proved.

Since  $L(\tau)$  is monotone decreasing on  $[0, \infty)$  and  $k < d$ , we obtain

$$(4.6) \quad L(\tau) > L(4) = \int_{|x| < 1/2} |x|^{-k} dx = C_2,$$

for any  $\tau \in (0, 4)$ . In the case of  $\lambda > \lambda_0$ , let  $\tau \in (0, T_w(\lambda))$ . Noting that  $0 < \tau < T_w(\lambda) \leq 4$ , by (4.1) and (4.6), we can get

$$\lambda < C_1 C_2^{-1} |\Re\mu|^{\frac{p-2}{p-1}} \tau^{(k+2)/2-q} = C_3 \tau^{-\kappa},$$

where  $C_3 := C_1 C_2^{-1} |\Re \mu|^{\frac{p-2}{p-1}} > 0$  and  $\kappa = (k+2)/2 - q$ . Since  $\kappa > 0$  due to  $k < 2/(p-1)$ , we have

$$\tau \leq C_4 \lambda^{-1/\kappa}.$$

Since  $\tau$  is arbitrary in  $(0, T_w(\lambda))$ , this implies  $T_w(\lambda) \leq C_4 \lambda^{-1/\kappa}$ , which completes the proof of the proposition.  $\square$

Before proving Theorem 2.3, we prepare the following lemma.

**Proposition 4.1.** *Let  $d \in \mathbb{N}$ ,  $s = 0, 1$  or  $s \in (0, \min(1, d/2))$  and  $1 \leq p \leq p_s$ . If  $u$  is a  $H^s$ -solution, then  $u$  becomes the weak-solution in the sense of Definition 2.1.*

We can prove the proposition in the similar manner as the proof of Proposition 3.1 in [28] for  $L^2$ -solution.

Now we give a proof of Theorem 2.3.

*Proof of Theorem 2.3.* Let  $\tau \in (0, T(\lambda))$  and  $u$  be a  $H^s$ -solution on  $[0, \tau)$ . By using Proposition 4.1,  $u$  also becomes the weak-solution to (NLS). By the assumption  $p \leq p_s$  and  $k < d/2 - s$ ,  $k < d/2 - s \leq \min(d, \frac{2}{p-1})$ , we can apply Proposition 2.2 to obtain

$$\tau \leq C \lambda^{-\kappa},$$

for  $\lambda > \lambda_0$ , which implies  $T(\lambda) \leq C \lambda^{-\kappa}$ . The blow-up of the norm  $\|u(t)\|_{H^s}$  (if  $p < p_s$ ) or  $\|u\|_{Y_{\gamma, \rho}^s(t)}$  (if  $p = p_s$ ) for the local  $H^s$ -solution follows from Theorems 1.3, 1.4, which completes the proof of the theorem.  $\square$

Next we prove Proposition 2.4.

*Proof of Proposition 2.4.* We only consider the case  $\Re \mu \neq 0$ . The other case can be treated in the similar manner. By Lemma 3.2, we have

$$(4.7) \quad \lambda \leq C_1 |\Re \mu|^{\frac{p-2}{p-1}} \tau^{(k+2)/2-q} \{L(\tau)\}^{-1},$$

for any  $\tau \in (0, T)$ , where  $L(\tau)$  is defined by (4.2). Since  $L(\tau)$  is monotone decreasing on  $[0, \infty)$  and  $k < d$ , we obtain

$$(4.8) \quad L(\tau) > L(4) = \int_{|x| < 1/2} |x|^{-k} dx =: C_2 (< \infty),$$

for any  $\tau \in (0, 4)$ . Thus by (4.7)-(4.8), we can get

$$(4.9) \quad 0 \leq \lambda \leq C_1 C_2^{-1} |\Re \mu|^{\frac{p-2}{p-1}} \tau^{(k+2)/2-q},$$

for any  $\tau \in (0, \min(4, T))$ . By the assumption  $2/(p-1) < k$ , we have  $(k+2)/2 - q > 0$ . Therefore, taking the limit  $\tau \rightarrow +0$  in (4.9), we can conclude  $\lambda = 0$ , which completes the proof of the proposition.  $\square$

Finally we give a proof of Theorem 2.5.

*Proof of Theorem 2.5.* By  $p > p_s$ , we have  $2/(p-1) < d/2 - s$ . Thus we can see that there exists a  $H^s$ -function  $f$  satisfying (2.2) with  $2/(p-1) < k < \min(d, d/2 - s)$ . Since we can apply Proposition 2.4, we can get the conclusion of the theorem.  $\square$

## 5. EXAMPLE

In this section, we give an example of the initial function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying the assumptions in Theorems 2.3, 2.5.

**Example 5.1.** Let  $d \in \mathbb{N}$ ,  $-d/2 < s$  and  $k < \min(d/2 - s, d)$ . If we put

$$(5.1) \quad f(x) := 2|x|^{-k}\chi(|x|),$$

where  $\chi \in C_0^\infty([0, \infty))$  satisfies  $0 \leq \chi(r) \leq 1$  for  $r \geq 0$ ,  $\chi(r) = 1$  on  $[0, 1]$ , and  $\chi(r) = 0$  for  $r \geq 2$ , then  $f$  belongs to  $H^s(\mathbb{R}^d)$  and satisfies

$$(5.2) \quad f(x) \geq \begin{cases} |x|^{-k}, & (|x| \leq 1), \\ 0, & (|x| > 1). \end{cases}$$

This fact can be verified as follows. We note that  $\min(d, d/2 - s) = d/2 - s$  by  $s > -d/2$ , and  $f$  belongs to  $C^\infty(\mathbb{R}^d \setminus \{0\})$  and satisfies (5.2).

Case 1.  $-\frac{d}{2} < s \leq 0$ . In this case, Let  $\frac{1}{\eta} = \frac{1}{2} - \frac{s}{d} \in (\frac{1}{2}, 1)$ . By the Sobolev embedding, we have

$$\|f\|_{H^s}^\eta \leq C\|f\|_{L^\eta}^\eta = C \int_{|x| \leq 1} |x|^{-\eta k} dx + C < \infty,$$

due to  $k < d/2 - s$ .

Case 2.  $0 < s$ . In this case, let  $\frac{1}{\nu} := \frac{d+2([s]+1-s)}{2d}$ , where  $[a]$  denotes the largest integer not exceeding  $a$ . By the Sobolev embedding, we have

$$\|f\|_{H^s}^\nu \leq C\|f\|_{W^{[s]+1, \nu}}^\nu \leq C \sum_{|\alpha| \leq [s]+1} \|\nabla^\alpha f\|_{L^\nu}^\nu \leq C \int_{|x| \leq 1} |x|^{-\nu(k+[s]+1)} dx + C < \infty,$$

by using  $k < d/2 - s$  again.

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*E-mail address:* mikedamath.kyoto-u.ac.jp

*E-mail address:* inui@math.kyoto-u.ac.jp

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, KYOTO UNIVERSITY. KITASHIRAKAWA-OIWAKECHO, SAKYO-KU, KYOTO 606-8502, JAPAN